JOURNAL OF ALGEBRA 138, 121–136 (1991)

Primitive Local Galois Representations: An Example

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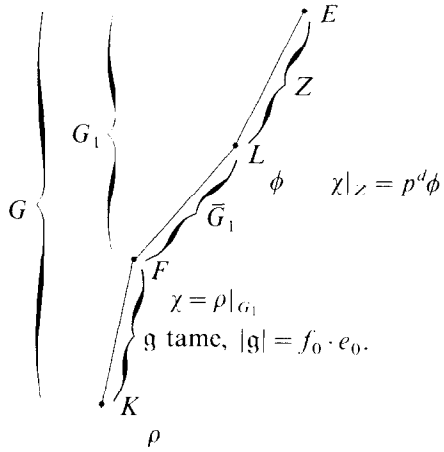
Communicated by A. Fröhlich

Received October 10, 1988

0. INTRODUCTION

The aim of this paper is to study the (Swan) conductor of primitive characters of local Galois groups.

Let $K/\mathbb{Q}_p$ be a p-adic number field, G_K its absolute Galois group, and ρ a primitive, irreducible, complex character of G_K. Let E/K be the fixed field of the kernel of ρ, $G := G_K/\ker \rho = \mathrm{Gal}(E/K)$, and let $E/F/K$ be the field corresponding to the wild ramification subgroup of G, $G_1 := \mathrm{Gal}(E/F)$. Let Z be the center of G_1 and L the corresponding field:



It follows from the work of Koch [4] that

- $\chi := \rho|_{G_1}$ is irreducible,
- Z is central in G,

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0021-8693/91 $3.00

— $\chi|_Z = p^d \cdot \phi$, where $\phi \in \operatorname{Irr} Z$ and $p^d = \chi(1)$,

— $\bar{G}_1 := G_1/Z$ is elementary Abelian with order p^{2d},

— the commutator induces a non-degenerate, alternating form

$$\langle\ ,\ \rangle: \bar{G}_1 \times \bar{G}_1 \to \mathbb{C}^*: \langle \bar{x}, \bar{y} \rangle := \phi([x, y]),$$

where $x, y \in G_1$, and $\bar{x}, \bar{y}$ denote their images in $\bar{G}_1$,

— this form is invariant under conjugation with elements of $\mathfrak{g} := \mathrm{Gal}(F/K)$, thus $\bar{G}_1$ becomes a symplectic $\mathbb{F}_p\mathfrak{g}$-module,

— $\bar{G}_1$ is an anisotropic $\mathbb{F}_p\mathfrak{g}$-module; i.e., it has no isotropic submodules.

The character ρ is called "in general position" if its conductor is minimal among the conductors of all twists $\rho\tau$, where τ runs through the one-dimensional characters of G_K.

If ρ is in general position, the Swan conductor of a twist is given by $\mathrm{sw}(\rho\tau) = \min\{\mathrm{sw}(\rho), \rho(1)\,\mathrm{sw}(\tau)\}$; see Lemma 1.6 below. Thus it suffices to determine the conductor of characters in general position.

If the group $\bar{G}_1$ has just one jump at $s = 1$, then Zink has computed the possible values of the conductor of primitive characters [11]. They have the form

$$\mathrm{sw}(\rho) = p^d \cdot \frac{1}{e_0} \cdot s \cdot (1 + p^{-v}), \tag{0.1}$$

where $p^d = \rho(1)$ is the character degree, $e_0 = e(F/K)$ is the tame ramification index of G, s is the jump of $\bar{G}_1$ ($s = 1$ in Zink's work), and v is an integer between 1 and d. Zink associates v with the degree of a polynomial belonging to the symplectic form on $\bar{G}_1$.

The same formula holds for general s if the degree of ρ is $\rho(1) = p$ [1]. In this case $\bar{G}_1$ is irreducible as $\mathbb{F}_p\mathfrak{g}$-module, and since ramification subgroups are invariant, i.e., submodules, $\bar{G}_1$ can have just one jump. Of course $v = 1$ in this case.

The method of Buhler applies to any primitive character ρ if its conductor is prime to p (and $\bar{G}_1$ has just one jump). We get $\mathrm{sw}(\rho) = p^d \cdot (1/e_0) \cdot s \cdot (1 + p^{-d})$ in this case.

Thus we have the question does Zink's formula (0.1) hold in general, i.e., if s is any integer such that $\bar{G}_1$ has just one jump at s? This was asked by Zink [10, 8.4, Bemerkung e].

However, I construct an example where formula (0.1) does not hold. In this example the jump of $\bar{G}_1$ is $s = 2$, the character degree is $\rho(1) = p^2$, and the conductor is $\mathrm{sw}(\rho) = p^2 \cdot (\frac{1}{5}) \cdot (2 + p^{-1})$. p is any prime with $p \equiv 2$

(mod 5), and the ground field is any p-adic number field that contains fifth roots of unity. The tame extension F/K has degree 5 and is totally ramified. The group G_1 is irreducible as $\mathbb{F}_p\mathfrak{g}$-module.

1. Preliminaries

I use the notation in the Introduction. But if I want to talk about general p-adic number fields I use the symbol R for the ground field and ω for a character of its Galois group.

The first six lemmas are either well known or straight forward. But I want to recall them in the form that I need here. Lemmas 1.7 and 1.8 are the crucial points in the construction of the example.

Lemma 1.1. *Let R_2/R_0 be a totally ramified Galois extension of p-adic number fields, $\mathrm{Gal}(R_2/R_1)$ be the wild ramification subgroup, and $\mathfrak{g} = \mathrm{Gal}(R_1/R_0)$ be the tame factor. Set $e_0 := |\mathfrak{g}|$. Let t be any integer, $t \geqq 1$.*

Then $U^t_{R_2}/U^{t+1}_{R_2}$ is a homogeneous $\mathbb{F}_p\mathfrak{g}$-module, i.e., isomorphic to several copies of one irreducible module V. To determine V, choose $\sigma \in \mathrm{Gal}(R_2/R_0)$ such that its image in $\mathfrak{g}$ generates $\mathfrak{g}$, and choose a prime element π of R_2.

Then $\pi^{\sigma-1} \equiv \omega \pmod{U^1_{R_2}}$, where ω is a primitive e_0th root of unity, and V can be identified with $\mathbb{F}_p(\omega^t)$, where σ acts by multiplication with ω^t.

Notation. $V = (\mathbb{F}_p(\omega^t),\ \sigma \to \omega^t)$.

Proof. For any $\tau \in \mathrm{Gal}(R_2/R_0)$ and $x = 1 + \pi^t\varepsilon \in U^t_{R_2}$, $x^{\tau-1} \equiv 1 + \pi^t \cdot (\pi^{\tau-1})^t \cdot \varepsilon \pmod{U^{t+1}_{R_2}}$. Thus the wild ramification subgroup acts trivial, and under the isomorphism

$$U^t_{R_2}/U^{t+1}_{R_2} \to \bar{R}_2 = \bar{R}_0$$
$$1 + \pi^t\varepsilon \to \bar{\varepsilon},$$

we can identify $U^t_{R_2}/U^{t+1}_{R_2}$ with a field extension of $\mathbb{F}_p$, where σ acts by multiplication with ω^t. That ω is a root of unity follows from the structure of the units of R_2 (see [2, Chap. 15]). Q.E.D.

Lemma 1.2. *Let ω be an irreducible complex character of the absolute Galois group G_R of the p-adic number field R. Then the Swan conductor of ω is given by*

$$\mathrm{sw}(\omega) = \omega(1) \cdot j,$$

where j is the largest index in the upper numbering of the ramification subgroups of G_R such that G^j_R is not contained in the kernel of ω.

Proof [6, Chap. VI, Sect. 2, Exercise 2]. Serre considers the Artin conductor $\mathrm{ar}(\omega)$. For an irreducible character ω, $\omega \neq 1$, the Swan conductor is defined by $\mathrm{sw}(\omega) = \mathrm{ar}(\omega) - \omega(1)$. Q.E.D.

LEMMA 1.3. *Using the notation in the Introduction, we have*

(i) $\mathrm{sw}(\rho) = (1/e_0)\,\mathrm{sw}(\chi)$,

(ii) *ρ is in general position iff χ is in general position.*

Proof. (i) follows from Lemma 1.2, as does (ii) "$\Rightarrow$". For "$\Leftarrow$" one must show that if $\chi\tau$ is in general position, with a one-dimensional character τ of G_F, then there exists a one-dimensional character τ' of G_K such that $\mathrm{sw}(\chi\tau) = \mathrm{sw}(\rho\tau' | G_F)$. The proof follows the idea of Buhler [1, Claim 1, p. 28]; see also Henniart [3, Theorems 1.7 and 5.3]. Q.E.D.

LEMMA 1.4. *Using the notation in the Introduction, if $\bar{G}_1$ has just one jump at s, then the conductor of ρ is of the form*

$$\mathrm{sw}(\rho) = \rho(1) \cdot \frac{1}{e_0} \cdot (s + a \cdot p^{\,v}),$$

with integers a and v such that $1 \leqq v \leqq d$, $1 \leqq a \leqq s$, and $\gcd(a, p) = 1$.

Proof [10, Proposition 8.4]. (Zink computes the possible values of $\mathrm{sw}(\chi)/\chi(1)$. Apply Lemmas 1.2 and 1.3 to Zink's formula.) Q.E.D.

Remark. The nature of v and a is not yet understood. Zink asked whether we always have $a = s$, but this is false, as the example shows.

LEMMA 1.5. *Let $L/F/K$ be Galois extensions of the p-adic number field K, let F/K be tamely, and let L/F be totally and wildly ramified with $\bar{G}_1 := \mathrm{Gal}(L/F)$ elementary Abelian.*

Assume that $\langle\ ,\ \rangle\colon \bar{G}_1 \times \bar{G}_1 \to \mathbb{C}^*$ is a non-degenerate, alternating form, invariant under $\mathfrak{g} := \mathrm{Gal}(F/K)$.

(i) *If $p \neq 2$, there exists a projective, irreducible representation $\bar{\rho}$ of G_K with kernel $\geqq G_L$ such that its restriction $\bar{\chi} := \bar{\rho}|_{G_F}$ remains irreducible and such that it induces the given form.*

Let ρ be any ordinary, complex character that lifts $\bar{\rho}$, and let $\rho|_{G_L} = \rho(1) \cdot \phi$, with ϕ a one-dimensional character of G_L. Then

$$\langle \bar{x}, \bar{y} \rangle = \phi([x, y]) \qquad \text{for} \quad x, y \in G_F,$$

where $\bar{x}$, $\bar{y}$ denote their images in $\bar{G}_1$.

(ii) *The character ρ of* (i) *is primitive iff the $\mathbb{F}_p\mathfrak{g}$-module $\bar{G}_1$ is anisotropic. In this case the projective representation $\bar{\rho}$ is unique.* (*And it exists even if $p=2$.*)

(iii) *The projective representation $\bar{\rho}$ of* (i) *is also unique if $|\mathfrak{g}|$ is prime to p, in particular if $|\mathfrak{g}|=1$, i.e., $F=K$.*

Proof. (i) [9, Satz 5]. (Note that any projective representation is liftable, since $H^2(G_K, \mathbb{C}^*)=0$; see [5, Part II].)

(ii) [9, Satz 6].

(iii) [9, Satz 3 (II)]. Q.E.D.

Lemma 1.5 allows us to define the conductor of an alternating form:

DEFINITION. Let L/F be an elementary Abelian and a totally and wildly ramified extension. Let

$$X: \mathrm{Gal}(L/F)\times\mathrm{Gal}(L/F)\to\mathbb{C}^*$$

be an alternating form. Define $j(X)$ as the minimum of $\mathrm{sw}(\chi)/\chi(1)$, where χ runs through the lifts of the projective representation $\bar{\chi}$ of G_F defined by X.

(To define $j(X)$ one just needs L/F to be Abelian; see [11]. But I just need those forms in the definition above.)

The next two lemmas give a simple observation about the conductor of the product of characters resp. forms:

LEMMA 1.6. *Let ω_1, ω_2 be two irreducible characters of the absolute Galois group of the p-adic field R such that $\omega_1\cdot\omega_2$ is irreducible too. Then*

$$\frac{\mathrm{sw}(\omega_1\cdot\omega_2)}{\omega_1(1)\cdot\omega_2(1)}\leqq\max\left\{\frac{\mathrm{sw}(\omega_1)}{\omega_1(1)},\frac{\mathrm{sw}(\omega_2)}{\omega_2(1)}\right\}.$$

And if $\mathrm{sw}(w_1)/\omega_1(1)\neq\mathrm{sw}(\omega_2)/\omega_2(1)$, we have equality in the formula above.

Proof. Use Lemma 1.2. Q.E.D.

LEMMA 1.7. *Let L/F be an elementary Abelian and a totally and wildly ramified extension of p-adic number fields, and let X_1, X_2 be two alternating forms on* $\mathrm{Gal}(L/F)$. *Then*

$$j(X_1\cdot X_2)\leqq\max\{j(X_1), j(X_2)\}.$$

And if $j(X_1)\neq j(X_2)$, equality holds in the formula above.

Proof. Set $X_3 := X_1 \cdot X_2$. Choose characters χ_1, χ_2, χ_3 of G_F such that χ_i belongs to X_i (in the sense of Lemma 1.5) and such that $j(X_i) = \mathrm{sw}(\chi_i)/\chi_i(1)$. "Belonging to X_i" means that $\chi_i|_{G_L} = \chi_i(1) \cdot \phi_i$, where ϕ_i is a one-dimensional character of G_L such that for $x, y \in G_F$,

$$X_i(\bar{x}, \bar{y}) = \phi_i([x, y]).$$

Hence

$$\phi_3|_{[G_F, G_F]} = \phi_1 \cdot \phi_2|_{[G_F, G_F]},$$

and thus

$$\phi_3 = \phi_1 \cdot \phi_2 \cdot \tau, \tag{*}$$

where τ is a one-dimensional character of G_L that is trivial on $[G_F, G_F]$.

Claim 1. $j_i := j(X_i) \overset{!}{=} \Phi_{L/F}(\mathrm{sw}(\phi_i))$, where $\Phi_{L/F}$ denotes the Herbrand function of L/F.

Proof of Claim 1. By Lemma 1.2, $G_F^{j_i} \not\leqq \ker \chi_i$, but for any $\varepsilon > 0$, $G_F^{j_i + \varepsilon} \leqq \ker \chi_i$. It follows that $G_F^{j_i + \varepsilon} \cap G_L \leqq \ker \phi_i$, and the claim asserts that $G_F^{j_i} \cap G_L \not\leqq \ker \phi_i$. Assume to the contrary that $G_F^{j_i} \cap G_L \leqq \ker \phi_i$. Then $G_F^{j_i} \cdot G_L$ is contained in the radical of X_i, because for $x \in G_F^{j_i}$ and $y \in G_F$, $[x, y] \in G_F^{j_i} \cap G_L$, and thus $X_i(\bar{x}, \bar{y}) = \phi_i([x, y]) = 1$. Now extend ϕ_i to a character $\tilde{\phi}_i$ of $G_F^{j_i} \cdot G_L$ such that $G_F^{j_i} \leqq \ker \tilde{\phi}_i$. The form X_i is also defined by $\tilde{\phi}_i$. Take any irreducible character $\tilde{\chi}_i$ of G_F lying over $\tilde{\phi}_i$. Then $\tilde{\chi}_i$ belongs to X_i, but $G_F^{j_i} \leqq \ker \tilde{\chi}_i$. This contradicts the minimality of $\mathrm{sw}(\chi_i)/\chi_i(1)$. Claim 1 follows.

The lemma now follows from

Claim 2. $\mathrm{sw}(\phi_3) \leqq \max\{\mathrm{sw}(\phi_1), \mathrm{sw}(\phi_2)\}$, with equality if $\mathrm{sw}(\phi_1) \neq \mathrm{sw}(\phi_2)$.

Proof of Claim 2. The form X_3 is defined by $\phi_1 \cdot \phi_2$. Since $\mathrm{sw}(\chi_3)/\chi_3(1)$ is chosen minimal, we get (by Claim 1 and Lemma 1.6)

$$\mathrm{sw}(\phi_3) \leqq \mathrm{sw}(\phi_1 \cdot \phi_2) \leqq \max\{\mathrm{sw}(\phi_1), \mathrm{sw}(\phi_2)\}. \tag{**}$$

Now consider the case $\mathrm{sw}(\phi_1) > \mathrm{sw}(\phi_2)$. By (*) and Lemma 1.6,

$$\mathrm{sw}(\phi_3) \leqq \max\{\mathrm{sw}(\phi_1 \cdot \tau), \mathrm{sw}(\phi_2)\}.$$

By the choice of χ_1 (and Claim 1) we know that $\mathrm{sw}(\phi_1 \cdot \tau) \geqq \mathrm{sw}(\phi_1)$. Thus by Lemma 1.6, $\mathrm{sw}(\phi_3) = \mathrm{sw}(\phi_1 \cdot \tau) \geqq \mathrm{sw}(\phi_1)$. Together with (**) this yields the equality. Q.E.D.

The following lemma is a variation of a result of Zink. I need some

Notation. If L/F is a Galois extension and $U \leqq L^*$ a subgroup of the multiplicative group of L, set

$$J_F U := \langle u^{\sigma-1} / u \in U, \sigma \in \mathrm{Gal}(L/F) \rangle.$$

LEMMA 1.8. *Let L/F be an Abelian extension of p-adic number fields such that* $\mathrm{Gal}(L/F)$ *has just one jump at s, $s \geqq 1$, $p \nmid s$. Let t be an integer with $t > s$. On $J_F U_L^t / J_F U_L^{t+1}$ consider the filtration*

$$V^j := (U_L^j \cap J_F U_L^t) \cdot J_F U_L^{t+1} / J_F U_L^{t+1}, \qquad j = 0, 1, \ldots.$$

The jumps *of this filtration occur at the indices*

$$j_\nu := t + p^\nu \cdot s, \qquad \nu = 0, \ldots, \min\{m-1, w-\delta\},$$

where m, w, and δ are defined as

$$p^m := |L:F|,\ w := \textit{p-value of } (t-s), \textit{ and if } s + (t-s)/p^w \equiv 0 \pmod{p}, \textit{ then } \delta := 1; \textit{ otherwise } \delta := 0.$$

Moreover the order *of the jumps is $|V^{j_\nu} : V^{j_\nu+1}| = q := |\bar{F}|$, where $\bar{F}$ denotes the residue class field.*

Remarks. (1) Zink considers the case $t = s$ [10, Proposition 8.5]. The proof is similar but differs in some technical details. The result for $t = s$ is more complicated.

(2) I need this lemma with $m = 2d$, $s = 1$, $t = \Psi_{L/F}(2) = 1 + p^{2d}$. In this case $w = m = 2d$ and $\delta = 0$.

Proof.

Claim 1. If $p \mid t$, then $J_F U_L^t = J_F U_L^{t+1}$.

Let $x \in U_L^t$ and $\sigma \in \mathrm{Gal}(L/F)$. To show that $x^{\sigma-1} \in (U_L^{t+1})^{\sigma-1}$, choose a prime element π_σ of the fixed field of σ, and write

$$x = (1 + \pi_\sigma^{t/p} \cdot \omega) \cdot x_1,$$

where ω is a p-regular root of unity (and thus in F) and where $x_1 \in U_L^{t+1}$. It follows that $x^{\sigma-1} = x_1^{\sigma-1}$.

Claim 2. Assume that $p \nmid t$. Then the first jump is $j_0 = s + t$, and its order is q.

From [6, Chap. IV, Sect. 2, Exercise 3a], for $x = 1 + y \in U_L^t$ and $\sigma \in \mathrm{Gal}(L/F)$, $\sigma \neq 1$ (and π a prime element of L) follows:

$$x^{\sigma-1} \equiv 1 + t \cdot (\pi^{\sigma-1} - 1) \cdot y \pmod{U_L^{s+t+1}}.$$

Thus $J_F U_L^t \leqq U_L^{s+t}$, and analogously $J_F U_L^{t+1} \leqq U_L^{s+t+1}$; moreover, for fixed σ, $\sigma \neq 1$, we get an isomorphism:

$$(U_L^t)^{\sigma-1}/(U_L^{t+1})^{\sigma-1} \xrightarrow{\sim} U_L^{s+t}/U_L^{s+t+1}. \qquad (***)$$

Hence the first jump is indeed j_0, and its order is given by

$$|J_F U_L^t : (J_F U_L^t \cap U_L^{s+t+1}| = |U_L^{s+t} : U_L^{s+t+1}|.$$

Claim 3. Assume that $p \nmid t$ and $|L:F| = p$. Then j_0 is the only jump.

This follows because $\mathrm{Gal}(L/F)$ is cyclic, thus $J_F U_L^t = (U_L^t)^{\sigma-1}$ for a generator σ of $\mathrm{Gal}(L/F)$. Now look at $(***)$ in the proof of Claim 2.

Claim 4. Assume that $p \nmid t$. The lemma follows by induction on m.

After Claim 3 we are left with the case $m > 1$. Choose an intermediate field $L/L'/F$ with $|L:L'| = p$. Consider the following sequences for $j \geqq s+t+1$:

$$U_L^j \cap J_{L'} L^* \rightarrowtail U_L^j \cap J_F U_L^t \xrightarrow{N_{L/L'}} U_{L'}^{\Phi_{L/L'}(j)} \cap J_F U_{L'}^{\Phi_{L/L'}(t)},$$

$$U_L^j \cap J_{L'} L^* \rightarrowtail U_L^j \cap J_F U_L^{t+1} \xrightarrow{N_{L/L'}} U_{L'}^{\Phi_{L/L'}(j)} \cap J_F U_{L'}^{\Phi_{L/L'}(t+1)}.$$

These sequences are exact. It suffices to consider the first one:

Surjectivity of the Norm. We have $N_{L/L'}(J_F U_L^t) = J_F(N_{L/L'} U_L^t) = J_F U_{L'}^{\Phi_{L/L'}(t)}$ and $N_{L/L'}(U_L^j) = U_{L'}^{\Phi_{L/L'}(j)}$, since j and t are greater then the jump s of L/L'; see [6, Chap. V, Sect. 6, Corollary 4 to Proposition 9]. Let $r \in U_{L'}^{\Phi_{L/L'}(j)} \cap J_F U_{L'}^{\Phi_{L/L'}(t)}$ with

$$r = N_{L/L'}(x), \qquad x \in U_L^j,$$

and

$$r = N_{L/L'}(y), \qquad y \in J_F U_L^t.$$

Then

$$\begin{aligned} h := xy^{-1} &\in \ker(N_{L/L'}) \cap U_L^j \cdot J_F U_L^t \\ &= J_{L'} L^* \cap U_L^j \cdot J_F U_L^t \qquad \text{(since } L/L' \text{ is cyclic)} \\ &\leqq J_{L'} L^* \cap U_L^{s+t} \qquad \text{(Claim 4).} \end{aligned}$$

But $J_{L'} L^* = (L^*)^{\sigma-1}$ for a generator σ of $\mathrm{Gal}(L/L')$, and $(L^*)^{\sigma-1} \cap U_L^{s+t} = (U_L^t)^{\sigma-1}$. This follows again from [6, Chap. IV, Sect. 2, Exercise 3a]. Thus $h \in (U_L^t)^{\sigma-1} \leqq J_F U_L^t$, and hence $x = yh \in J_F U_L^t \cap U_L^j$.

Exactness in the Middle. The kernel of the norm is

$$\begin{aligned} U_L^j \cap J_F U_L^t \cap \ker(N_{L/L'}) &= U_L^j \cap J_F U_L^t \cap J_{L'} L^* \\ &= U_L^j \cap J_F U_L^t \cap J_{L'} U_L^t \\ &\qquad \text{(since } U_L^j \text{ and } J_F U_L^t \text{ are in } U_L^{s+t}\text{)} \\ &= U_L^j \cap J_{L'} U_L^t = U_L^j \cap J_{L'} L^*. \end{aligned}$$

Consider the cokernel of the embedding of the second sequence into the first one. This yields an isomorphism:

$$\frac{U_L^j \cap J_F U_L^t}{U_L^j \cap J_F U_L^{t+1}} \xrightarrow[\simeq]{N_{L/L'}} \frac{U_{L'}^{\Phi_{L/L'}(j)} \cap J_F U_{L'}^{\Phi_{L/L'}(t)}}{U_{L'}^{\Phi_{L/L'}(j)} \cap J_F U_{L'}^{\Phi_{L/L'}(t+1)}}. \qquad (****)$$

Thus apart from the jump j_0, the jumps of $J_F U_L^t / J_F U_L^{t+1}$ are determined by the right side of (****). We have $\Phi_{L/L'}(t) = s + (t-s)/p$. If $t \not\equiv s \pmod p$, then $\Phi_{L/L'}(t)$ is not an integer, and the right side of (****) vanishes. If $t \equiv s \pmod p$, but $\Phi_{L/L'}(t) \equiv 0 \pmod p$, the right side vanishes too (Claim 3). In all other cases the jumps of the right side are given by induction: Let $|L' : F| = p^{m'}$ with $m' = m - 1$. $\mathrm{Gal}(L'/F)$ has just one jump at s. Set $t' := \Phi_{L/L'}(t)$. The p-value of $(t' - s)$ is $w' = w - 1$, and the summand δ is the same as before. Thus the jumps of the right side are given by

$$j'_\nu = t' + p^\nu \cdot s, \qquad \nu = 0, \ldots, \min\{m' - 1, w' - \delta\},$$

and the order of all jumps is q.

Hence $J_F U_L^t / J_F U_L^{t+1}$ has the jump j_0 and the jumps

$$\Psi_{L/L'}(j'_\nu) = t + p^{\nu+1} \cdot s, \qquad \nu + 1 = 1, \ldots, \min\{m - 1, w - \delta\}.$$

$\Psi_{L/L'}$ denotes the inverse of the Herbrand function $\Phi_{L/L'}$. Q.E.D.

2. Example

In the notation of the introduction choose an odd prime p such that $p \equiv 2 \pmod 5$. As ground field choose a p-adic number field that contains fifth roots of unity. Let F/K be a totally and tamely ramified Galois extension of degree $e_0 := 5$. To do this, choose a prime element π_K of K and set $\pi_F := \sqrt[5]{\pi_K}$. Then $F = K(\pi_F)$ and π_F is a prime element of F. $\mathfrak{g} := \mathrm{Gal}(F/K) = \langle \sigma \rangle$ with $\pi_F^\sigma = \alpha \cdot \pi_F$, where α is a primitive fifth root of unity in K.

PROPOSITION 2.1. *There exists a primitive character ρ of G_K with the following properties:*

(i) *Let E be the fixed field of the kernel of ρ. Then E is a totally and wildly ramified extension of F.*

(ii) *Let $Z = \mathrm{Gal}(E/L)$ be the center of $\mathrm{Gal}(E/K)$. Then L is an extension of F, and $\bar{G}_1 := \mathrm{Gal}(L/F)$ is a chief factor of $\mathrm{Gal}(L/K)$, i.e., $\bar{G}_1$ is an irreducible $\mathbb{F}_p\mathfrak{g}$-module.*

(iii) $p^d := \rho(1) = p^2$.

(iv) *$\bar{G}_1$ has just one jump at $s = 2$.*

(v) *ρ is in general position.*

But

(vi) $\mathrm{sw}(\rho) = p^d \cdot (1/e_0) \cdot (2 + p^{-1}) = p \cdot (2p + 1)/5$.

Remark 2.2. There also exist primitive characters ρ' satisfying conditions (i)–(v), but whose conductor is prime to p; i.e.,

$$\mathrm{sw}(\rho') = p^d \cdot \frac{1}{e_0} \cdot (2 + 2 \cdot p^{-d}) = \frac{2(p^2 + 1)}{5}.$$

After the proof of Proposition 2.1 I sketch the proof of the remark.

Proof. If ρ' is any primitive character satisfying (i)–(v), then by Lemma 1.4 the conductor has the form

$$\mathrm{sw}(\rho') = p^d \cdot \frac{1}{e_0} \cdot (s + a \cdot p^{-v}),$$

$$\text{with } 1 \leqq a \leqq s = 2 \text{ and } 1 \leqq v \leqq d = 2.$$

But $\mathrm{sw}(\rho')$ is an integer! Hence there are only two possibilities:

(I) $v = 2$ and $a = 2$: $\mathrm{sw}(\rho') = p^d \cdot (1/e_0) \cdot (s + s \cdot p^{-2})$,

(II) $v = 1$ and $a = 1$: $\mathrm{sw}(\rho') = p^d \cdot (1/e_0) \cdot (s + 1 \cdot p^{-1})$.

I must show that the second possibility does occur!

Step 1. I construct a primitive character ρ' satisfying (i)–(v): Consider U_F^2/U_F^3 as $\mathbb{F}_p\mathfrak{g}$-module. By Lemma 1.1 it is homogenous with irreducible submodule $V = (\mathbb{F}_p(\alpha^2),\ \sigma \to \alpha^2)$. Since e_0 is odd, $\mathbb{F}_p(\alpha^2) = \mathbb{F}_p(\alpha)$, and $\dim_{\mathbb{F}_p} V = 2d = 4$.

Now we need an alternating form on V. I write forms multiplicatively, but the description of the forms on V is better done additively. Thus let me fix

once and for all an embedding $\mathbb{F}_p^+ \to \mathbb{C}^*$. The $\mathfrak{g}$-invariant, alternating forms on V are given as follows: Let $\phi_0 \in \mathrm{Gal}(\mathbb{F}_p(\alpha)/\mathbb{F}_p)$ with $\mathrm{ord}\,\phi_0 = 2$. Then every $\mathfrak{g}$-invariant form Y is given by an element $v_0 \in \mathbb{F}_p(\alpha)$ with $\phi_0(v_0) = -v_0$:

$$Y(v, w) = \operatorname{tr}_{\mathbb{F}_p(\alpha)/\mathbb{F}_p}(v_0 \cdot v \cdot \phi_0(w)) \to \mathbb{C}^*.$$

Here $\operatorname{tr}_{\mathbb{F}_p(\alpha)/\mathbb{F}_p}$ dentoes the trace. This form is $\mathfrak{g}$-invariant since $\phi_0(\alpha) = \alpha^p = \alpha^{-1}$; see [7, Theorem 3.1]. Choose a $\mathfrak{g}$-invariant subgroup N^2, $U_F^3 \leqq N^2 \leqq U_F^2$, such that $U_F^2/N^2 \simeq V$ as $\mathbb{F}_p\mathfrak{g}$-module. Further choose a $\mathfrak{g}$-invariant subgroup $N' \leqq F^*$ such that $U_F^2 \cdot N' = F^*$ and $U_F^2 \cap N' = N^2$. Let L'/F be the Abelian extension such that $N_{L'/F}L'^* = N'$ (this is local class field theory; see [6, Chap. XIV, Sect. 6, Theorem 1]):

$$V \simeq \begin{cases} U_F^2 \\ N^2 \\ U_F^3 \end{cases} \begin{array}{l} \\ \\ \\ \end{array} \begin{array}{l} F^* \\ N' = N_{L'/F}L'^* \\ \\ \end{array}$$

Hence $\mathrm{Gal}(L'/F) \simeq V$ as $\mathbb{F}_p\mathfrak{g}$-module and $\mathrm{Gal}(L'/F)$ has a $\mathfrak{g}$-invariant, alternating form X'.

By Lemma 1.5 there exists a projective representation $\bar{\rho}'$ of G_K belonging to the form X', and a suitable lifting of $\bar{\rho}'$ gives us a primitive character ρ' that satisfies (i)–(v), with L' as the fixed field of the center of $G_K/\ker \rho'$.

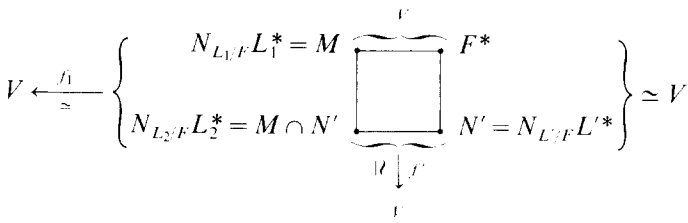
Step 2. As remarked at the beginning of the proof, there are two possibilities for $\mathrm{sw}(\rho')$. If $\mathrm{sw}(\rho') = p^2 \cdot \frac{1}{5} \cdot (2 + p^{-1})$, we are done. Thus let us assume that this is not the case. Then $\mathrm{sw}(\rho') = p^2 \cdot \frac{1}{5} \cdot (2 + 2 \cdot p^{-2})$, and I must construct a new character ρ.

The center of ρ will belong to a new field L/F. To find this field, consider U_F^1/U_F^2 as $\mathbb{F}_p\mathfrak{g}$-module. By Lemma 1.1 it is homogeneous with irreducible submodule $(\mathbb{F}_p(\alpha), \sigma \to \alpha)$. But since $p \equiv 2 \pmod{e_0}$, the map

$$(\mathbb{F}_p(\alpha), \sigma \to \alpha) \to (\mathbb{F}_p(\alpha), \sigma \to \alpha^2) = V$$
$$x \longrightarrow x^p$$

is an $\mathbb{F}_p\mathfrak{g}$-module isomorphism.

Choose a $\mathfrak{g}$-invariant subgroup N^1, $U_F^2 \leqq N^1 \leqq U_F^1$, and a $\mathfrak{g}$-invariant subgroup M with $U_F^1 \cdot M = F^*$ and $U_F^1 \cap M = N^1$. Let L_1/F be the Abelian extension with $N_{L_1/F}L_1^* = M$, and let L_2/F be that extension with $N_{L_2/F}L_2^* = M \cap N'$:



Fix isomorphisms

$$f_1 : \mathrm{Gal}(L_2/L_1) \xrightarrow{\sim} V, \qquad f' : \mathrm{Gal}(L_2/L') \xrightarrow{\sim} V,$$

and

$$f_1 \oplus f' : \mathrm{Gal}(L_2/F) \xrightarrow{\sim} V \oplus V.$$

Fix a $\mathfrak{g}$-invariant, alternating form Y on V that corresponds to the form X' under $\mathrm{Gal}(L'/F) \xrightarrow{\sim} \mathrm{Gal}(L_2/L_1) \xrightarrow{f_1} V$. On $V \oplus V$ we get the non-degenerate form

$$Y \times Y : (V \oplus V) \times (V \oplus V) \to \mathbb{C}^*$$
$$(v_1 \oplus w_1, v_2 \oplus w_2) \to Y(v_1, v_2) \cdot Y(w_1, w_2).$$

We also have the form $Y \times 1$ with radical $0 \oplus V$,

$$Y \times 1 : (V \oplus V) \times (V \oplus V) \to \mathbb{C}^*$$
$$(v_1 \oplus w_1, v_2 \oplus w_2) \to Y(v_1, v_2),$$

and similarly the form $1 \times Y$ with radical $V \oplus 0$.

To construct the form that will belong to the character ρ, consider the diagonal

$$D^+ := \{v \oplus v / v \in V\}$$

and the "anti-diagonal"

$$D^- := \{v \oplus (-v) / v \in V\}.$$

The form $Y \times Y$ is non-degenerate on both diagonals, and they are orthogonal to each other with respect to this form.

I want to construct two forms Y_+ and Y_- with radicals D^+ and D^-, resp., such that $Y_+|_{D^- \times D^-} = Y \times Y|_{D^- \times D^-}$, and similarly for Y_-. This is done by setting

$$Y_+(v_1 \oplus w_1, v_2 \oplus w_2) := Y(v_1 - w_1, v_2 - w_2)^{1/2}.$$

Note that the square root is unique within pth roots of unity.

The form that defines the character ρ will correspond to Y_+^2. Read all these forms on $\mathrm{Gal}(L_2/F)$ by the fixed isomorphism. Let $\mathrm{Gal}(L_2/L) \leq \mathrm{Gal}(L_2/F)$ be the group corresponding to D^+, and denote by X the form on $\mathrm{Gal}(L/F)$ corresponding to the form Y_+^2 on $(V \oplus V)/D^+$. By Lemma 1.5 we get a primitive character ρ that satisfies (i)–(v).

Step 3. It remains to compute the conductor of ρ.

Lemma 1.3 reduces this problem to the conductor $\chi := \rho|_{G_F}$, and by Lemma 1.5 we can translate this into the conductor of the form X.

Let X_2 be the form on $\mathrm{Gal}(L_2/F)$ that corresponds to $Y \times Y$ on $V \oplus V$, let X_1 correspond to $1 \times Y$, and let X' correspond to $Y \times 1$. Then $X_2 = X_1 \cdot X'$ with conductors

$$j(X') = 2 + 2 \cdot p^{-2} \qquad \text{by assumption on } \mathrm{sw}(\rho'),$$

and

$$j(X_1) = 1 + 1 \cdot p^{-1} \text{ (By Lemma 1.4,} \qquad j(X_1) = 1 + p^{-v}, \text{ with } v = 1, 2.$$

But X_1 is g-invariant, thus the conductor of ρ_1 of G_K belonging to X_1 is an integer. Hence $j(X_1)$ is "divisible" by $e_0 = 5$.) Hence Lemma 1.7 implies that

$$j(X_2) = 2 + 2 \cdot p^{-2}.$$

Define a form Z on $\mathrm{Gal}(L_2/F)$ by

$$X_2 = X \cdot Z.$$

I show that $j(Z) > j(X_2)$. Then by Lemma 1.7, $j(X) = j(Z)$.

It remains to prove the following

Claim. $j(Z) = 2 + p^{-1}$.

Thus let us study the form Z:

$$\mathrm{Gal}(L_2/L_1) \text{ is isotropic for } Z. \tag{*1}$$

This I compute on $V \oplus V$. Restrict $Y \times Y = Y_+^2 \cdot Z$ to $(V \oplus 0) \times (V \oplus 0)$:

$$Y_+^2(v_1 \oplus 0, v_2 \oplus 0) = Y(v_1, v_2) = Y \times Y(v_1 \oplus 0, v_2 \oplus 0).$$

Thus $Z(v_1 \oplus 0, v_2 \oplus 0) = 1$.

It follows that there exists a one-dimensional character θ of G_{L_1} that defines Z: By Lemma 1.5 we get a character χ_Z of G_F with $\chi_Z|_{G_{L_2}} = \chi_Z(1) \cdot \phi_Z$, and ϕ_Z defines Z. But G_{L_1} is isotropic for Z; this means that $G_{L_1}/\ker \phi_Z$ is Abelian, and thus ϕ_Z is extendible to θ on G_{L_1}.

Choose θ such that its conductor is minimal. Then

$$j(Z) = \Phi_{L_1/F}(\mathrm{sw}(\theta)). \qquad (*2)$$

This follows by the same argument as the Claim 1 in the proof of Lemma 1.7. It remains to show that

Claim'. $\mathrm{sw}(\theta) \overset{!}{=} \Psi_{L_1/F}(2 + p^{-1}) = 1 + p^4(1 + p^{-1}) = 1 + p^4 + p^3$. ($\Psi_{L_1/F}$ denotes the inverse of the Herbrand function. Note that $|L_1/F| = p^4$ and that L_1/F has just one jump at $s_1 = 1$.)

Translate the problem into the relative Weil group $W := W_{L_1/F}$:

$$\begin{array}{ccccc}
L_1^* & \rightarrowtail & W & \twoheadrightarrow & \mathrm{Gal}(L_1/F) \\
\downarrow & & \downarrow & & \| \\
G_{L_1}^{ab} & \rightarrowtail & G_F/G'_{L_1} & \twoheadrightarrow & \mathrm{Gal}(L_1/F) \\
\| & & & & \\
G_{L_1}/G'_{L_1} & & & &
\end{array}$$

(W is the Weilgroup of L_1^{ab}/F, a subgroup of $\mathrm{Gal}(L_1^{ab}/F) = G_F/G'_{L_1}$. See [8, Appendices II and III.) Then Z becomes a $\mathfrak{g}$-invariant form on W, θ a character of L_1^*, and

$$Z(x, y) = \theta([x, y]) \qquad \text{for} \quad x, y \in W.$$

The relative Weil group is the context in which to formulate the next two facts:

$$U_{L_1}^{\Psi_{L_1/F}(2)+1} \text{ is contained in the radical of } Z. \qquad (*3)$$

This follows since $N_{L_1/F}(U_{L_1}^{\Psi_{L_1/F}(2)+1}) = U_F^3 \leqq M \cap N'$; see [6, Chap. V, Sect. 6, Corollary 4 to Proposition 9]. But $M \cap N' = N_{L_2/F}(L_2^*)$ is contained in the radical of Z. Here I identified $F^*/M \cap N'$ with $\mathrm{Gal}(L_2/F)$ by local class field theory.

$$U_{L_1}^{\Psi_{L_1/F}(2)} \text{ is not contained in the radical of } Z. \qquad (*4)$$

Assume that (*4) is false. Again read Z as a form on F^* by class field theory. Then $N_{L_1/F}(U_{L_1}^{\Psi_{L_1/F}(2)}) = U_F^2$ is contained in the radical of Z (again see [6, Chap. V, Sect. 6, Corollary 4 to Proposition 9]). But also $M \cap N' \leqq \mathrm{rad}\, Z$. Thus $U_F^2 \cdot (M \cap N') = M \leqq \mathrm{rad}\, Z$. Now transport this problem to $V \oplus V$. Then $Y \times Y = Y_+^2 \cdot Z$ and $V \oplus 0 \leqq \mathrm{rad}\, Z$, $D^+ \leqq \mathrm{rad}\, Y_+^2$. This implies that $V \oplus 0$ and D^+ are orthogonal to each other (with respect to

$Y \times Y$). But this is nonsense: The orthogonal complement of $V \oplus 0$ is $0 \oplus V$ and that of D^+ is D^-.

Translate (*3) and (*4) into a fact about θ. Restrict θ to $[U_{L_1}^{\Psi_{L_1/F}(2)}, W] = J_F U_{L_1}^{\Psi_{L_1/F}(2)}$: This restriction is not trivial by (*4), but $J_F U_{L_1}^{\Psi_{L_1/F}(2)+1}$ is contained in the kernel of θ by (*3). Set $t := \Psi_{L_1/F}(2) = 1 + p^4$. It follows from Lemma 1.8 (with $s = 1$) that $\theta|_{J_F U_{L_1}^t}$ has a "conductor" of the form $j_v = t + p^v \cdot 1$, $v = 0, 1, 2, 3$. That means that one can write $\theta = \theta_0 \cdot \theta_1$ with $\mathrm{sw}(\theta_0) = j_v$ and $\theta_1|_{J_F U_{L_1}^t} = 1$. Thus it suffices to show

Claim". $v = 3$; i.e., $j_v = t + p^3 = 1 + p^4 + p^3$.

(From Claim" only $\mathrm{sw}(\theta) \geqq j_3$ follows. But $\mathrm{sw}(\theta)$ cannot be greater, otherwise $j(Z) = j(X)$ would be greater and thus $sw(\rho)$ would be greater as possible.) The reason for Claim" is that Z is $\mathfrak{g}$-invariant. Thus $\theta|_W$ is $\mathfrak{g}$-invariant and in particular:

> θ restricted to $(U_{L_1}^{j_v} \cap J_F U_{L_1}^t)$ is $\mathfrak{g}$-invariant and not trivial, but θ restricted to $(U_{L_1}^{j_v+1} \cap J_F U_{L_1}^t)$ becomes trivial.

Hence we can read θ as a $\mathfrak{g}$-invariant character of a subgroup of $U_{L_1}^{j_v}/U_{L_1}^{j_v+1}$. But the last group is a homogeneous $\mathbb{F}_p\mathfrak{g}$-module with irreducible submodule $V = (\mathbb{F}_p(\alpha^{j_v}), \sigma \to \alpha^{j_v})$, where α is a primitive fifth root of unity (Lemma 1.1). Since θ is $\mathfrak{g}$-invariant and one-dimensional, V must be one-dimensional too. Thus the order of α^{j_v} is a divisor of $p-1$. Since $\mathrm{ord}(\alpha) = 5$ and $p \equiv 2 \pmod 5$, it follows that $j_v \equiv 0 \pmod 5$. Thus $j_v = 1 + p^4 + p^3$. Q.E.D.

Sketch of Proof of Remark 2.2. Take the symplectic $\mathbb{F}_p\mathfrak{g}$-module $V := U_F^2/N^2$ constructed in Step 1 of the Proof of Proposition 2.1. Denote the symplectic form on V by X.

Claim. Choose any Abelian Extension M/F that has just one jump at $s = 2$ and such that $N_{M/F}U_M^2$ is maximally isotropic for X. Then one can extend X to an alternating form X' on F^* such that $N_{M/F}M^*$ is isotropic for X' and such that $j(X') = 2(1 + p^{-2})$ if and only if $d(X) = f - 2$.

Here $f = |\bar{F} : \mathbb{F}_p|$ and the definition of $d(X)$ is as follows: Identity $U_F^2/U_F^3 = \bar{F}$. Denote by τ the Frobenius of $\bar{F}/\mathbb{F}_p$. Then there is an alternating polynomial $P = \sum_{i=1}^{f-1} a_i \cdot \tau^i$ such that $X(x, y) = \mathrm{tr}_{\bar{F}/\mathbb{F}_p}(x \cdot P(y))$. (See Zink [11, Sect. 3] for the theory of alternating polynomials.) Denote by $d(X)$ the degree of P.

Since X is $\mathfrak{g}$-invariant, the degree of P is indeed $f - 2$: $\mathfrak{g}$ acts by multiplication with α^2, a fifth root of unity. Hence

$$P = \alpha^2 \cdot P \cdot \alpha^2 = \sum_{i=1}^{f-1} a_i \alpha^2 \cdot \tau^i(\alpha^2) \cdot \tau^i = \sum_{i=1}^{f-1} a_i \alpha^{2+2p^i} \cdot \tau^i.$$

Thus $a_i \neq 0$ only for those i for which $\alpha^{p^i} = \alpha^{-1}$, that is, for even i. Hence $d(x)$ is even. On the other hand, $|U_F^2 : N^2| = p^4$; thus $f > d(X) \geqq f - 2$; see [11, Sect. 4, Proposition 6 (iv)]. Since f contains fifth roots of unity and $p \equiv 2 \pmod 5$, f is even too. This implies that $d(X) = f - 2$.

The claim follows from a generalization of a result of Zink to an arbitrary jump s ([11, Sect. 3, Proposition 9]). Zink already formulated most of the steps of the proof of his Proposition 9 for general s. One just needs the right extension of his definition of $j(X)$ to extend the result of Proposition 9.

The form X' constructed in the claim is not necessarily $\mathfrak{g}$-invariant. Define a $\mathfrak{g}$-invariant form by $Y_0(x, y) := \prod_{\sigma \in \mathfrak{g}} X'(\sigma(x), \sigma(y))$. Y_0 on U_F^2 equals $X^{|\mathfrak{g}|}$. Since $|\mathfrak{g}|$ is prime to p, choose an integer c such that $c \cdot |\mathfrak{g}| \equiv 1 \pmod p$. Let $Y := Y_0^c$. Then Y is a $\mathfrak{g}$-invariant extension of X and $j(Y) \leqq 2(1 + p^{-1})$.

Assume that Y is not the form we are looking for. Then $U_1 := U_F^2 \cdot \mathrm{Rad}(Y) < F^*$, but $U_1/\mathrm{Rad}(Y)$ is Y-non-degenerate. Let U_2 be the orthogonal complement of U_1. Write $Y = Y_1 \cdot Y_2$, with $\mathrm{Rad}(Y_i) = U_i \cdot Y_1$ and Y_2 are $\mathfrak{g}$-invariant forms. Y_1 is a form with jumps $\leqq 1$. The work of Zink implies that $j(Y_1) = 1 + p^{-2}$. Y_2 is a form that has just one jump at $s = 2$. Hence $j(Y_2)$ equals $2(1 + p^{-2})$ or $2 + p^{-1}$. These conductors are all different, hence $j(Y_2) = j(Y)$. Since $j(Y)$ is small, $j(Y_2)$ is small too, and Y_2 is the form we were looking for.

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